



Soft Set Theory Applied to Hoops

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Abstract

In this paper, we introduced the concept of a soft hoop and we investigated some of their properties. Then, we established different types of intersections and unions of the family of soft hoops. We defined two operations \odot and \rightarrow on the set of all soft hoops and we proved that with these operations, it is a hoop and also is a Heyting algebra. Finally we introduced a congruence relation on the set of all soft hoops and we investigated the quotient of it.

1 Introduction

Hoops are naturally ordered commutative residuated integral monoids, introduced by B. Bosbach in [11, 12] then study by J. R. Büchi and T. M. Owens in [13], a paper never published. In the last years, hoops theory was enriched with deep structure theorems(see [2, 4, 5, 7, 8, 9, 10, 15, 23]). Many of these results have a strong impact with fuzzy logic. Particularly, from the structure theorem of finite basic hoops one obtains an elegant short proof of the completeness theorem for propositional basic logic, introduced by Hájek in [17]. The algebraic structures corresponding to Hájek's propositional (fuzzy) basic logic, BL-algebras, are particular cases of hoops. The main example of BL-algebras in interval $[0, 1]$ endowed with the structure induced by a t-norm. MV-algebras, product algebras and Gödel algebras are the most known classes of BL-algebras. Recent investigations are concerned with non-commutative generalizations for these structures.

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On the other hand, soft set theory was initiated by Molodtsov [21] in 1999 as a new mathematical tool for modelling the uncertainties arising from the parametrization of elements of a universe. He mentioned several directions for the applications of soft sets. In fact, before soft set theory, there have been some mathematical theories such as probability theory, fuzzy set theory, rough set theory, vague set theory, and interval mathematics theory for dealing with uncertainties. However, the superiority of the soft set theory compared with other mathematical tools, is its ability of parametrization. Aktas and Cagman [3] compared soft sets to the related concepts of fuzzy sets and rough sets. They also introduced the notion of soft groups. After them, soft algebraic structures have been studied by many authors [14, 18, 19, 20, 24, 25, 26]. Specially, Y. B. Jun [18] introduced and investigated soft BCK/BCI-algebras. Moreover, by combination of fuzzy set theory with soft set theory, fuzzy soft algebraic structures were born. For example, Hadipour et al. [16] defined the notion of fuzzy soft BF-algebra and investigated the level subset, union and intersection, fuzzy soft image and fuzzy soft inverse image of them. The most soft algebraic structures are defined as follows: for a set of parameters E and a general algebra X , a pair (F, E) is called a soft general algebra over X if F is a map of E into the set of all subsets of the set X such that for each $e \in E$, $F(e)$ is the empty set or a subalgebra of X .

In this paper, the concept of a soft hoop is introduced and some examples are provided. Then, different types of intersections and unions of the family of soft hoops are established. We defined two operations \odot and \rightarrow on the set of all soft hoops on a hoop H and a parameters set E and we proved that with these operations is a hoop and also is a Heyting algebra. Finally we introduced a congruence relation on the set of all soft hoops and we investigated the quotient of it.

2 Preliminaries

In this section, we recollect some definitions and results which will be used in the following and we shall not cite them every time they are used.

Molodtsov ([21]) defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. We denote $\rho(U)$ the power set of U and $A \subseteq E$. A pair (λ, A) is called a *soft set* over U , where λ is a map given by $\lambda : A \rightarrow \rho(U)$. In other words, a soft set over U is a parametrized family of subsets of the universe U . For $\varepsilon \in A$, $\lambda(\varepsilon)$ may be considered as the set of ε -approximate elements of the soft set (λ, A) . Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [21].

Let (F, A) and (G, B) be two soft sets over U . Then,

(i) (F, A) is said to be a *soft subset* of (G, B) , denoted by $(F, A) \subseteq (G, B)$, if

$A \subseteq B$ and $F(a) \subseteq G(a)$, for all $a \in A$.

(ii) Two soft sets (F, A) and (G, B) are said to be a *soft equal*, denoted by $(F, A) = (G, B)$, if $(F, A) \subseteq (G, B)$ and $(G, B) \subseteq (F, A)$.

Let U be an initial universe and E be the set of parameters such that $A \subseteq E$. Then,

(i) (F, A) is called a *relative null soft set* (with respect to the parameter set A), denoted by ϕ_A , if $F(a) = \phi$, for all $a \in A$.

(ii) (G, A) is called a *relative whole soft set* (with respect to the parameter set A), denoted by U_A , if $G(a) = U$, for all $a \in A$.

The relative whole soft set with respect to the set of parameters E is called the *absolute soft set* over U and simply denoted by U_E . In a similar way, the relative null soft set with respect to E is called the *null soft set* over U and is denoted by \emptyset_E . We shall denote by \emptyset_E the unique soft set over U with an empty parameter set, which is called the *empty soft set* over U . Note that \emptyset_\emptyset and \emptyset_E are different soft sets over U and $\emptyset_\emptyset \subseteq \emptyset_A \subseteq (F, A) \subseteq U_A \subseteq U_E$, for all soft set (F, A) over U (See [21, 22]).

The next definition introduces three types of intersections and three types of unions of the family of soft sets over a common universe set. For a family $\{(F_i, A_i) \mid i \in I\}$ of soft sets over U , we give some definitions as below:

- (a) The *extended intersection* of the family $\{(F_i, A_i) \mid i \in I\}$ is defined as the soft set $\bigcap_{i \in I} (F_i, A_i) = (H, C)$, where $C = \bigcup_{i \in I} A_i$ and $H(x) = \bigcap_{i \in I(x)} F_i(x)$; $I(x) = \{i \mid x \in A_i\}$, for all $x \in C$.
- (b) The *restricted intersection* of the family (F_i, A_i) is defined as the soft set $\prod_{i \in I} (F_i, A_i) = (H, C)$, where $C = \bigcap_{i \in I} A_i \neq \emptyset$ and $H(x) = \bigcap_{i \in I} F_i(x)$, for all $x \in C$.
- (c) The *extended union* of the family (F_i, A_i) is defined as the soft set $\bigsqcup_{i \in I} (F_i, A_i) = (H, C)$, where $C = \bigcup_{i \in I} A_i$, $H(x) = \bigcup_{i \in I(x)} F_i(x)$, and $I(x) = \{i \mid x \in A_i\}$, for all $x \in C$.
- (d) The *restricted union* of the family (F_i, A_i) is defined as the soft set $\bigcup_{i \in I} (F_i, A_i) = (H, C)$, where $C = \bigcap_{i \in I} A_i \neq \emptyset$ and $H(x) = \bigcup_{i \in I} F_i(x)$, for all $x \in C$.
- (e) The \wedge -*intersection* of the family (F_i, A_i) is defined as the soft set $\bigwedge_{i \in I} (F_i, A_i) = (H, C)$, where $C = \prod_{i \in I} A_i$ and $H((a_i)_{i \in I}) = \bigcap_{i \in I} F_i(a_i)$, for all $(a_i)_{i \in I} \in C$.

- (f) The \vee -intersection of the family (F_i, A_i) is defined as the soft set
- $$\bigvee_{i \in I} (F_i, A_i) = (H, C), \text{ where } C = \prod_{i \in I} A_i \text{ and } H((a_i)_{i \in I}) = \bigcup_{i \in I} F_i(a_i),$$
- for all $(a_i)_{i \in I} \in C$.

A hoop is an algebra $(H, \odot, \rightarrow, 1)$ of type $(2, 2, 0)$ such that, for all $x, y, z \in H$,

(HP1) $(H, \odot, 1)$ is a commutative monoid,

(HP2) $x \rightarrow x = 1$,

(HP3) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$,

(HP4) $x \odot (x \rightarrow y) = y \odot (y \rightarrow x)$.

On hoop H we define $x \leq y$ if and only if $x \rightarrow y = 1$. It is easy to see that " \leq " is a partial order relation on H . A hoop H is *bounded* if there is an element $0 \in A$ such that $0 \leq x$, for all $x \in A$. Let $x^0 = 1$, $x^n = x^{n-1} \odot x$, for any $n \in \mathbb{N}$. If H is a bounded hoop, then we define a negation " $'$ " on H by, $x' = x \rightarrow 0$, for all $x \in H$. If $(x')' = x$, for all $x \in H$, then the bounded hoop H is said to have the *double negation property*, or (DNP) for short (See [15]).

The following proposition provides some properties of hoop.

Proposition 2.1. [11, 12] *Let $(H, \odot, \rightarrow, 1)$ be a hoop. Then the following conditions hold, for all $x, y, z \in H$,*

(i) (H, \leq) is a \wedge -semilattice with $x \wedge y = x \odot (x \rightarrow y)$,

(ii) $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$,

(iii) $x \odot y \leq x, y$ and $x \leq y \rightarrow x$,

(iv) $x \rightarrow 1 = 1$ and $1 \rightarrow x = x$,

(v) $x \leq y$ implies $x \odot z \leq y \odot z, z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$.

Proposition 2.2. [15] *Let H be a bounded hoop. Then the following conditions hold, for all $x, y \in H$,*

(i) $x \leq x''$ and $x''' = x'$,

(ii) $x \odot x' = 0$ and $x \leq x' \rightarrow y$.

Let H be a hoop. A non-empty subset F of H is called a *filter* of H if it satisfies the following assertion,

(F1) $x, y \in F$ implies $x \odot y \in F$,

(F2) $x \in F$ and $x \leq y$ imply $y \in F$, for any $y \in H$.

Clearly, $1 \in F$, for all filters F of H . A filter F of H is called a *proper filter* if $F \neq H$. It can be easily seen that, if H is a bounded hoop, then a filter is proper if and only if it is not containing 0 (See [15]).

Note. From now on, in this paper, $(H, \odot, \rightarrow, 1)$ or simply H is a hoop and E is a set of parameters, unless otherwise state.

3 Soft sub-hoops

In this section, we introduce the notions of sub-hoop and soft-hoop, and investigate some properties of them.

Definition 3.1. A *sub-hoop* of a hoop H is a subset S of H containing the unit element of H . It means that a subset S of a hoop H is called a *sub-hoop* of H if $1 \in S$, $x \odot y \in S$ and $x \rightarrow y \in S$, for all $x, y \in S$.

Example 3.2. (i) Every filter F of H is a sub-hoop of H .

Because, for any $x, y \in F$, $x \odot y \in F$ and by Proposition 2.1(iii), $y \leq x \rightarrow y$, since $y \in F$ and F is a filter of H , $x \rightarrow y \in F$.

(ii) Let $H = \{0, a, b, c, d, 1\}$ be a set. Define the operations \odot and \rightarrow on H as follow:

\rightarrow	0	a	b	c	d	1	\odot	0	a	b	c	d	1
0	1	1	1	1	1	1	0	0	0	0	0	0	0
a	c	1	b	c	b	1	a	0	a	d	0	d	a
b	d	a	1	b	a	1	b	0	d	c	c	0	b
c	a	a	1	1	a	1	c	0	0	c	c	0	c
d	b	1	1	b	1	1	d	0	d	0	0	0	d
1	0	a	b	c	d	1	1	0	a	b	c	d	1

By routine calculations we can see that $(H, \odot, \rightarrow, 0, 1)$ is a bounded hoop. Let $S_1 = \{a, 1\}$ and $S_2 = \{b, c, 1\}$. It is easy to see that S_1 and S_2 are two sub-hoops of H .

Proposition 3.3. Let $\{H_i \mid i \in I\}$ be a family of sub-hoops of H . Then $\bigcap_{i \in I} H_i$ is a sub-hoop of H .

Proof. The proof is clear. □

In the following example, we show that the union of a family of sub-hoops may not be a sub-hoop, in general.

Example 3.4. Let H be a hoop as in Example 3.2(ii). As we see, S_1 and S_2 are two sub-hoops of H . But it is clear that $S = S_1 \cup S_2 = \{a, b, c, 1\}$ is not a sub-hoop of H , because $a \odot b = d \notin S$.

In the following proposition we investigate that under which condition, the union of a family of sub-hoops is a sub-hoop.

Proposition 3.5. Let $\{H_i \mid i \in I\}$ be a family of sub-hoops of H . If, for any $i, j \in I$, $H_i \subset H_j$ or $H_j \subset H_i$, then $\bigcup_{i \in I} H_i$ is a sub-hoop of H .

Proof. The proof is clear. \square

Definition 3.6. Let (λ, A) be a soft set over H , where $A \subseteq E$. Then (λ, A) is called a *soft hoop* of H if $\lambda(e)$ is a sub-hoop of H , for any $e \in A$.

The set of all soft hoops over H is denoted by $\text{Soft}_E(H)$, that is

$$\text{Soft}_E(H) = \{(\lambda_i, A_j) \mid A_j \subseteq E, \lambda_i : A_j \rightarrow \rho(H) \text{ and } (\lambda_i, A_j) \text{ is a soft hoop of } H\}$$

Example 3.7. (i) Let $H = \{0, a, b, 1\}$ be a set. Define the operations \odot and \rightarrow on H as follows:

\rightarrow	0	a	b	1	\odot	0	a	b	1
0	1	1	1	1	0	0	0	0	0
a	a	1	1	1	a	0	0	a	a
b	0	a	1	1	b	0	a	b	b
1	0	a	b	1	1	0	a	b	1

Routine calculations show that $(H, \odot, \rightarrow, 0, 1)$ is a bounded hoop. Let $A \subseteq E$ in which $\{A_1, A_2\}$ is a partition of A . Define a set-valued function as follows:

$$\lambda : A \rightarrow \rho(H), e \mapsto \begin{cases} \{b, 1\} & \text{if } e \in A_1, \\ \{0, a, 1\} & \text{if } e \in A_2. \end{cases} \quad (3.1)$$

Then (λ, A) is a soft hoop over H .

(ii) Let $\mu : H \rightarrow [0, 1]$ be a map such that, for any $x, y \in H$, we have

$$\mu(x \odot y) \geq \min\{\mu(x), \mu(y)\} \quad \text{and} \quad \mu(x \rightarrow y) \geq \min\{\mu(x), \mu(y)\}.$$

For any $\varepsilon \in [0, 1]$, consider the set $U(\mu; \varepsilon) = \{x \in H \mid \mu(x) \geq \varepsilon\}$. Define a map

$$\lambda : [0, 1] \rightarrow \rho(H), \varepsilon \mapsto U(\mu, \varepsilon).$$

Then $(\lambda, [0, 1])$ is a soft hoop over H .

Proposition 3.8. Let (λ, A) be a soft hoop of H and $B \subseteq A$. Then $(\lambda|_B, B)$ is a soft hoop of H , too.

Proof. It is obvious. \square

In the following example, we show that the converse of Proposition 3.8 may not be true, in general.

Example 3.9. Let $H = \{0, a, b, c, d, 1\}$ be a bounded hoop as in Example 3.2(ii), then let $A \subseteq E$ such that $\{A_1, A_2\}$ be a partition of A . Define a set-valued function as follows:

$$\lambda : A \rightarrow \rho(H), e \mapsto \begin{cases} \{0, 1\} & \text{if } e \in A_1, \\ \{0, c, d, 1\} & \text{if } e \in A_2. \end{cases} \quad (3.2)$$

Then (λ, A_1) is a soft hoop over H , but (λ, A) is not a soft hoop over H . Because if $e \in A_2$, then $\lambda(e) = \{0, c, d, 1\}$ is not a sub-hoop of H . But if we take $B = A_1 \subset A$, then $(\lambda|_B, B)$ is a soft hoop over H .

Theorem 3.10. Let H be a hoop and $\{(\lambda_i, A_j)\}_{i,j \in I} \subseteq \text{Soft}_E(H)$. Then the following statements hold:

- (i) The extended intersection $\bigcap_{i,j \in I} (\lambda_i, A_j)$ is a soft hoop over H , if it is non-null.
- (ii) The restricted intersection $\prod_{i,j \in I} (\lambda_i, A_j)$ is a soft hoop over H , if it is non-null.
- (iii) The \bigwedge -intersection $\bigwedge_{i,j \in I} (\lambda_i, A_j)$ is a soft hoop over H , if it is non-null.

Proof. (i) As we know, $\bigcup_{j \in I} A_j \subset A$ and by Proposition 3.3, $\bigcap_{i \in I} \lambda_i$ is a sub-hoop of H . So $\bigcap_{i,j \in I} (\lambda_i, A_j)$ is a soft hoop over H .

The proof of other items is similar to the proof of (i). \square

In the following example, we show that different kinds of union of a family of soft hoops may not be a soft hoop, in general.

Example 3.11. Let H be a hoop as Example 3.2(ii) and $A = \{e_1, e_2\} \subseteq E$ such that $A_1 = \{e_1\}$ and $A_2 = \{e_2\}$. Define $\lambda_1(e_1) = \{a, 1\}$ and $\lambda_2(e_2) = \{b, c, 1\}$. Routine calculations show that (λ_1, A_1) and (λ_2, A_2) are two soft hoops over H , but $(\lambda_1, A_1) \cup (\lambda_2, A_2) = (\{e_1, e_2\}, \{a, b, c, 1\})$ is not a soft hoop over H . Because $a \odot b = d \notin \{a, b, c, 1\}$.

Definition 3.12. Let (λ, A) and (γ, B) be two soft hoops over H . Then (λ, A) is called a *soft sub-hoop* of (γ, B) if it satisfies:

- (i) $A \subseteq B$,
- (ii) $\lambda(e)$ is a sub-hoop of $\gamma(e)$, for all $e \in A$.

If (λ, A) is a soft sub-hoop of (γ, B) , then we denoted it by $(\lambda, A) \subseteq (\gamma, B)$. Two soft sub-hoops (λ, A) and (γ, B) are said to be a *soft equal hoop*, denoted by $(\lambda, A) = (\gamma, B)$, if $(\lambda, A) \subseteq (\gamma, B)$ and $(\gamma, B) \subseteq (\lambda, A)$.

Example 3.13. Let H be a hoop as Example 3.2 and $E = \{e_1, e_2, e_3\}$. Define

$$A = \{e_1\}, \gamma(e_1) = \{1, a\}, B = \{e_1, e_2\}, \lambda(e) = \begin{cases} \{0, a, c, 1\} & \text{if } e = e_1, \\ \{c, 1\} & \text{if } e = e_2. \end{cases}$$

It is clear that $(\gamma, A) \subseteq (\lambda, B)$.

Proposition 3.14. (i) If (λ, A) and (γ, A) are two soft hoops over H such that $\lambda(e)$ is a sub-hoop of $\gamma(e)$, for all $e \in A$, then (λ, A) is a soft sub-hoop of (γ, A) .

(ii) If (λ, H) and $(\varphi, \{1\})$ are two soft hoops over H , then $(\varphi, \{1\})$ is a soft sub-hoop of (λ, H) .

Proof. It is straightforward. \square

4 Algebraic structures on $\text{Soft}_E(H)$

In this section, we introduce some operations on $\text{Soft}_E(H)$ and we show that $\text{Soft}_E(H)$ with these operations is a Heyting algebra and a hoop. Also, we define a congruence relation on $\text{Soft}_E(H)$ and investigate the quotient that is made by it.

Note. From now on, in this section, for any $e \in A$, $\lambda(e) \tilde{\subseteq} \gamma(e)$ means that $\lambda(e)$ is a sub-hoop of $\gamma(e)$.

Theorem 4.1. $(\text{Soft}_E(H), \subseteq)$ is a poset.

Proof. Let $(\lambda, A) \in \text{Soft}_E(H)$. Since $A \subseteq A$ and for any $e \in A$, $\lambda(e) \tilde{\subseteq} \lambda(e)$, by Definition 3.12, we have $(\lambda, A) \subseteq (\lambda, A)$. Hence, \subseteq is reflexive. Now, let $(\lambda, A), (\gamma, B) \in \text{Soft}_E(H)$, such that $(\lambda, A) \subseteq (\gamma, B)$ and $(\gamma, B) \subseteq (\lambda, A)$. Then $A \subseteq B$ and, for any $e \in A$, $\lambda(e) \tilde{\subseteq} \gamma(e)$. Also, $B \subseteq A$ and for any $e \in B$, $\gamma(e) \tilde{\subseteq} \lambda(e)$. Hence, $(\lambda, A) = (\gamma, B)$, and so \subseteq is antisymmetry. Now, we prove that the relation \subseteq is transitive. Let $(\lambda, A), (\gamma, B), (\eta, C) \in \text{Soft}_E(H)$, such that $(\lambda, A) \subseteq (\gamma, B)$ and $(\gamma, B) \subseteq (\eta, C)$. Then $A \subseteq B$ and, for any $e \in A$, $\lambda(e) \tilde{\subseteq} \gamma(e)$ and $B \subseteq C$ and, for any $e \in B$, $\gamma(e) \tilde{\subseteq} \eta(e)$. It is easy to see that $A \subseteq C$ and $\lambda(e) \tilde{\subseteq} \eta(e)$, for any $e \in A$. Then $(\lambda, A) \subseteq (\eta, C)$, and so $(\text{Soft}_E(H), \subseteq)$ is a poset. \square

Note. In the following example, we show that extended union of the family of soft hoops is not a soft hoop, in general. So, from now on, in this section, we suppose that $\text{Soft}_E(H)$ is closed under two operations \coprod and \coprod .

Example 4.2. Let H be a bounded hoop as in Example 3.2(ii). Suppose $E = \{e_1, e_2, e_3, e_4\}$ and define

$$A_1 = \{e_1, e_2, e_3, e_4\} \quad , \quad \lambda_1(e) = \begin{cases} \{1\} & \text{if } e = e_1, \\ \{a, 1\} & \text{if } e = e_2, \\ \{0, 1\} & \text{if } e \in \{e_3, e_4\} \end{cases}$$

$$A_2 = \{e_1, e_2, e_3\} \quad , \quad \lambda_2(e) = \begin{cases} \{1, c\} & \text{if } e = e_3, \\ \{1, b, c\} & \text{if } e \in \{e_1, e_2\} \end{cases}$$

$$A_3 = \{e_1, e_2, e_3, e_4\} \quad , \quad \lambda_3(e) = \begin{cases} H & \text{if } e = e_1, \\ \{0, a, c, 1\} & \text{if } e = e_4, \\ \{1\} & \text{if } e \in \{e_3, e_2\} \end{cases}$$

$$A_4 = \{e_2, e_3\} \quad , \quad \lambda_4(e) = \begin{cases} \{0, 1\} & \text{if } e = e_3, \\ \{a, 1\} & \text{if } e = e_2. \end{cases}$$

By routine calculation, it is clear that (λ_i, A_i) , for $1 \leq i \leq 4$, are soft hoops of H . But $(\gamma, B) = \prod_{i=1}^4 (\lambda_i, A_i)$ is not a soft hoop of H . Because, for $e = e_3$, we have $\gamma(e) = \{0, c, 1\}$, which is not a sub hoop of H , since $c \rightarrow 0 = a \notin \gamma(e)$.

Definition 4.3. A soft set (λ, A) over H is called a *trivial soft hoop* (resp., *whole soft hoop*) if it is a soft hoop over H and satisfies the condition $\lambda(e) = \{1\}$ (resp., $\lambda(e) = H$), for all $e \in A$.

Theorem 4.4. $(\text{Soft}_E(H), \prod, \coprod, H_E, \emptyset_\emptyset)$ is a complete distributive bounded lattice, where $H_E = (H, E)$ is the whole soft hoop over H and $\emptyset_\emptyset = (\emptyset, \emptyset)$ is a null soft hoop over H .

Proof. By Theorem 4.1, $(\text{Soft}_E(H), \subseteq)$ is a poset. Also, we have

$$\emptyset_\emptyset \subseteq \emptyset_A \subseteq (\lambda, A) \subseteq H_A \subseteq H_E$$

Hence, \emptyset_\emptyset and H_E are the smallest and the greatest elements of $\text{Soft}_E(H)$, respectively, and so $\text{Soft}_E(H)$ is bounded.

Now, we prove that $(\text{Soft}_E(H), \prod, \coprod, H_E, \emptyset_\emptyset)$ is a lattice. Let $(\lambda, A), (\gamma, B), (\eta, C) \in \text{Soft}_H(E)$. Then it is clear that $(\lambda, A) \coprod (\lambda, A) = (\lambda, A)$. Also,

$$(\lambda, A) \prod (\gamma, B) = (\lambda \cup \gamma, A \cup B) = (\gamma \cup \lambda, B \cup A) = (\gamma, B) \prod (\lambda, A)$$

Moreover, we have

$$\begin{aligned}
(\lambda, A) \coprod((\gamma, B) \coprod(\eta, C)) &= (\lambda, A) \coprod(\gamma \cup \eta, B \cup C) \\
&= (\lambda \cup (\gamma \cup \eta), A \cup (B \cup C)) \\
&= ((\lambda \cup \gamma) \cup \eta, (A \cup B) \cup C) \\
&= (\lambda \cup \gamma, A \cup B) \coprod(\eta, C) \\
&= ((\lambda, A) \coprod(\gamma, B)) \coprod(\eta, C)
\end{aligned}$$

Finally,

$$\begin{aligned}
(\lambda, A) \coprod((\lambda, A) \coprod(\gamma, B)) &= (\lambda, A) \coprod(\lambda \cup \gamma, A \cup B) \\
&= (\lambda \cap (\lambda \cup \gamma), A \cap (A \cup B)) \\
&= (\lambda, A)
\end{aligned}$$

By the similar way, we can proof all these conditions for the operation \prod . Hence, $(\text{Soft}_E(H), \prod, \coprod)$ is a lattice. Also, it is clear that $(\text{Soft}_E(H), \prod, \coprod)$ is a complete lattice.

So, it is enough to prove that $(\text{Soft}_E(H), \prod, \coprod)$, is a distributive lattice. For this, we have

$$\begin{aligned}
(\lambda, A) \prod[(\gamma, B) \coprod(\eta, C)] &= (\lambda, A) \prod(\gamma \cup \eta, B \cup C) \\
&= [\lambda \cap (\gamma \cup \eta), A \cap (B \cup C)] \\
&= [(\lambda \cap \gamma) \cup (\lambda \cap \eta), (A \cap B) \cup (A \cap C)] \\
&= (\lambda \cap \gamma, A \cap B) \prod(\lambda \cap \eta, A \cap C) \\
&= [(\lambda, A) \prod(\gamma, B)] \prod[(\lambda, A) \prod(\eta, C)]
\end{aligned}$$

The proof of other case is similar. Therefore, $(\text{Soft}_E(H), \prod, \coprod, H_E, \emptyset_\emptyset)$ is a complete distributive bounded lattice. \square

Let $(\lambda, A), (\gamma, B) \in \text{Soft}_E(H)$. Then we define the operation \rightarrow on $(\text{Soft}_E(H), \prod, \coprod, H_E, \emptyset_\emptyset)$ as follows:

$$(\lambda, A) \rightarrow (\gamma, B) = \prod_{i,j \in I} \{(\lambda_i, A_j) \mid (\lambda, A) \prod(\lambda_i, A_j) \subseteq (\gamma, B)\}$$

Proposition 4.5. *Let $(\lambda, A), (\gamma, B) \in \text{Soft}_E(H)$. Then the following assertions, hold:*

- (i) $(\lambda, A) \rightarrow (\lambda, A) = H_E$.
- (ii) $(\lambda, A) \subseteq (\gamma, B)$ if and only if $(\lambda, A) \rightarrow (\gamma, B) = H_E$.

Proof. (i) Let $(\lambda, A), (\gamma, B) \in \text{Soft}_E(H)$. Then,

$$\begin{aligned}
& (\lambda, A) \rightarrow (\lambda, A) \\
&= \prod_{i,j \in I} \{(\lambda_i, A_j) | (\lambda, A) \prod (\lambda_i, A_j) \subseteq (\lambda, A)\} \\
&= \prod_{i,j \in I} \{(\lambda_i, A_j) | (\lambda \cap \lambda_i, A \cap A_j) \subseteq (\lambda, A)\} \\
&= \prod_{i,j \in I} \{(\lambda_i, A_j) | A \cap A_j \subseteq A \text{ and } (\lambda \cap \lambda_i)(e) \tilde{\subseteq} \lambda(e), \text{ for any } e \in A \cap A_j\} \\
&= \left(\bigcup_{i \in I} \lambda_i, \bigcup_{j \in I} A_j \right) \\
&= H_E
\end{aligned}$$

(ii) Let $(\lambda, A) \subseteq (\gamma, B)$. Then by Definition 3.12, $A \subseteq B$ and $\lambda(e) \tilde{\subseteq} \gamma(e)$, for any $e \in A$. Thus,

$$\begin{aligned}
(\lambda, A) \rightarrow (\gamma, B) &= \prod_{i,j \in I} \{(\lambda_i, A_j) | (\lambda, A) \prod (\lambda_i, A_j) \subseteq (\gamma, B)\} \\
&= \prod_{i,j \in I} \{(\lambda_i, A_j) | (\lambda \cap \lambda_i, A \cap A_j) \subseteq (\gamma, B)\} \\
&= H_E
\end{aligned}$$

Since $A \cap A_j \subseteq B$ and $(\lambda \cap \lambda_i)(e) \tilde{\subseteq} \gamma(e)$, for any $e \in A \cap A_j$.

Conversely, assume that $(\lambda, A) \rightarrow (\gamma, B) = H_E$. Then, for any $(\eta, C) \in \text{Soft}_E(H)$ we have, $(\lambda, A) \prod (\eta, C) \subseteq (\gamma, B)$. Hence, for $(\eta, C) = (\lambda, A)$, we have $(\lambda, A) \subseteq (\gamma, B)$. \square

Theorem 4.6. $(\text{Soft}_E(H), \prod, \rightarrow, H_E, \emptyset_\emptyset)$ is a bounded hoop.

Proof. The operation \prod will be consider as a product operation on $\text{Soft}_E(H)$. So

$$(\lambda, A) \prod (\gamma, B) = (\lambda \cap \gamma, A \cap B)$$

Then it is clear that $\text{Soft}_E(H)$ is closed with respect to \prod . Now, we prove that $(\text{Soft}_E(H), \prod, H_E)$ is a commutative monoid with the greatest element.

For this, for any $(\lambda, A), (\gamma, B), (\eta, C) \in \text{Soft}_E(H)$, we have,

$$\begin{aligned} (\lambda, A) \prod \{(\gamma, B) \prod (\eta, C)\} &= (\lambda, A) \prod (\gamma \cap \eta, B \cap C) \\ &= (\lambda \cap (\gamma \cap \eta), A \cap (B \cap C)) \\ &= ((\lambda \cap \gamma) \cap \eta, (A \cap B) \cap C) \\ &= (\lambda \cap \gamma, A \cap B) \prod (\eta, C) \\ &= \{(\lambda, A) \prod (\gamma, B)\} \prod (\eta, C) \end{aligned}$$

and

$$(\lambda, A) \prod H_E = (\lambda, A) \prod (H, E) = (\lambda \cap H, A \cap E) = (\lambda, A)$$

Therefore, $(\text{Soft}_E(H), \prod, H_E)$ is a commutative monoid with the greatest element H_E . By Proposition 4.5(i), $(\lambda, A) \rightarrow (\lambda, A) = H_E$,

$$\begin{aligned} &= (\lambda, A) \rightarrow ((\gamma, B) \rightarrow (\eta, C)) \\ &= \prod_{i,j \in I} \{(\lambda_i, A_j) | (\lambda_i, A_j) \cap (\lambda, A) \subseteq \prod_{k,l \in I} \{(\eta_k, C_l) | (\gamma, B) \prod (\eta_k, C_l) \subseteq (\eta, C)\}\} \end{aligned}$$

So we have, for every $i, j \in I$, $(\lambda_i, A_j) \cap (\lambda, A) \subseteq \prod_{k,l \in I} \{(\eta_k, C_l)\}$. Then,

$$(\lambda_i, A_j) \cap (\lambda, A) \cap (\gamma, B) \subseteq \prod_{k,l \in I} \{(\eta_k, C_l) \cap (\gamma, B)\}.$$

Since $(\text{Soft}_E(H), \prod, \prod)$ is a distributive lattice, we get that

$$(\lambda_i, A_j) \cap (\lambda, A) \cap (\gamma, B) \subseteq \prod_{k,l \in I} \{(\eta_k, C_l) \cap (\gamma, B)\} \subseteq (\eta, C).$$

Therefore,

$$\begin{aligned} &(\lambda, A) \rightarrow ((\gamma, B) \rightarrow (\eta, C)) \\ &= \prod_{i,j \in I} \{(\lambda_i, A_j) | (\lambda_i, A_j) \cap (\lambda, A) \cap (\gamma, B) \subseteq (\eta, C)\} \\ &= ((\lambda, A) \cap (\gamma, B)) \rightarrow (\eta, C) \end{aligned}$$

Now, it is enough to prove that,

$$(\lambda, A) \prod (\gamma, B) = (\lambda, A) \prod ((\lambda, A) \rightarrow (\gamma, B))$$

So, by Theorem 4.4, since $(\lambda, A) \prod (\gamma, B) = (\gamma, B) \prod (\lambda, A)$, we consequence that,

$$\begin{aligned} (\lambda, A) \prod ((\lambda, A) \rightarrow (\gamma, B)) &= (\lambda, A) \prod (\gamma, B) = (\gamma, B) \prod (\lambda, A) \\ &= (\gamma, B) \prod ((\gamma, B) \rightarrow (\lambda, A)) \end{aligned}$$

Thus, according to definition \rightarrow , it is clear that,

$$(\gamma, B) \subseteq \prod_{i,j \in I} \{(\lambda_i, A_j) | (\lambda, A) \prod (\lambda_i, A_j) \subseteq (\gamma, B)\} = (\lambda, A) \rightarrow (\gamma, B)$$

Then

$$(\lambda, A) \prod (\gamma, B) \subseteq (\lambda, A) \prod ((\lambda, A) \rightarrow (\gamma, B))$$

Conversely,

$$\begin{aligned} & (\lambda, A) \prod ((\lambda, A) \rightarrow (\gamma, B)) \\ = & (\lambda, A) \prod \left(\prod_{i,j \in I} \{(\lambda_i, A_j) | (\lambda, A) \prod (\lambda_i, A_j) \subseteq (\gamma, B)\} \right) \\ = & \prod_{i,j \in I} \{(\lambda_i \cap \lambda, A_j \cap A) | (\lambda \cap \lambda_i, A \cap A_j) \subseteq (\gamma, B)\} \end{aligned}$$

Since $(\lambda \cap \lambda_i)(e) \tilde{\subseteq} \gamma(e)$, we have $(\lambda \cap \lambda \cap \lambda_i)(e) \tilde{\subseteq} (\lambda \cap \gamma)(e)$, for any $e \in A \cap A_i$. Also, from $A \cap A_j \subseteq B$, we get that $A \cap A \cap A_i \subseteq A \cap B$. So,

$$\prod_{i \in I} (\lambda \cap \lambda_i)(e) \tilde{\subseteq} (\lambda \cap \gamma)(e) \quad , \quad \bigcup_{j \in I} (A \cap A_j) \subseteq A \cap B$$

Hence,

$$\prod_{i,j \in I} (\lambda \cap \lambda_i, A \cap A_j) \subseteq (\lambda \cap \gamma, A \cap B)$$

Finally,

$$\begin{aligned} & (\lambda, A) \prod ((\lambda, A) \rightarrow (\gamma, B)) \\ = & (\lambda, A) \prod \left(\prod_{i,j \in I} \{(\lambda_i, A_j) | (\lambda, A) \prod (\lambda_i, A_j) \subseteq (\gamma, B)\} \right) \\ \subseteq & (\lambda, A) \prod (\gamma, B) \end{aligned}$$

Therefore, $(\text{Soft}_E(H), \prod, \rightarrow, H_E, \emptyset_\emptyset)$ is a bounded hoop. \square

Definition 4.7. [6] A *Heyting algebra* is an algebra $(A, \vee, \wedge, \rightarrow, 1)$ of type $(2, 2, 2, 0)$, where $(A, \vee, \wedge, 1)$ is a lattice with the greatest element 1 and, for any $x, y, z \in A$,

$$x \leq y \rightarrow z \text{ if and only if } x \wedge y \leq z \quad (*)$$

Theorem 4.8. $(\text{Soft}_E(H), \prod, \prod, \rightarrow, H_E)$ is a Heyting algebra.

Proof. By Theorem 4.4, we prove that $(\text{Soft}_E(H), \prod, \coprod, H_E, \emptyset_\emptyset)$ is a distributive lattice with the greatest element H_E . Now, it is enough to prove that the condition (*) hold. Let (λ, A) , (γ, B) and $(\eta, C) \in \text{Soft}_E(H)$ such that $(\lambda, A) \subseteq (\gamma, B) \rightarrow (\eta, C)$. By definition of the operation \rightarrow , we have

$$(\gamma, B) \rightarrow (\eta, C) = \prod_{i,j \in I} \{(\mu_i, D_j) \mid (\mu_i, D_j) \cap (\gamma, B) \subseteq (\eta, C)\} = \prod_{i,j \in I} (\mu_i, D_j)$$

Since $(\lambda, A) \subseteq (\gamma, B) \rightarrow (\eta, C)$, we get that $(\lambda, A) \subseteq \prod_{i,j \in I} (\mu_i, D_j)$. Then

$$(\lambda, A) \prod (\gamma, B) \subseteq (\gamma, B) \prod (\prod_{i,j \in I} (\mu_i, D_j))$$

Moreover, since the lattice is distributive, we consequence that

$$(\lambda, A) \prod (\gamma, B) \subseteq \prod_{i,j \in I} [(\mu_i, D_j) \prod (\gamma, B)] \subseteq \prod (\eta, C) = (\eta, C)$$

So, $(\lambda, A) \prod (\gamma, B) \subseteq (\eta, C)$.

Conversely, suppose $(\lambda, A) \prod (\gamma, B) \subseteq (\eta, C)$. Then, according to definition of \rightarrow , we consequence that $(\lambda, A) \in (\gamma, B) \rightarrow (\eta, C)$. Hence, $(\lambda, A) \subseteq (\gamma, B) \rightarrow (\eta, C)$. Therefore, $(\text{Soft}_E(H), \prod, \coprod, \rightarrow, H_E)$ is a Heyting algebra. \square

As we know, by $(\rightarrow, \emptyset_\emptyset)$ we can define

$$(\lambda, A) \rightarrow \emptyset_\emptyset = \prod_{i,j \in I} \{(\lambda_i, A_j) \mid (\lambda_i, A_j) \cap (\lambda, A) \subseteq (\emptyset, \emptyset)\}.$$

Since for any $\emptyset_\emptyset \neq (\lambda_i, A_j) \in \text{Soft}_E(H)$, $1 \in \lambda_i$, then $(\lambda_i, A_j) \cap (\lambda, A) \neq (\emptyset, \emptyset)$. Hence, $(\lambda, A) \rightarrow \emptyset_\emptyset = \emptyset_\emptyset$. Therefore, we conclude that for any $(\lambda, A) \in \text{Soft}_E(H)$, $(\lambda, A) \rightarrow \emptyset_\emptyset = \emptyset_\emptyset$.

Example 4.9. Let H be a hoop in Example 3.7 and $E = \{e_1, e_2\}$. Then define the set of soft hoops as follows:

$A_1 = \emptyset$ and $\lambda_1 : A \rightarrow \rho(H)$ such that $\lambda_1 = \emptyset$ so $(\lambda_1, A_1) = (\emptyset, \emptyset) = \emptyset_\emptyset$,

$A_2 = E$ and $\lambda_2 : E \rightarrow \rho(H)$ such that $\lambda_2 = H$, so $(\lambda_2, A_2) = H_E$,

$A_3 = \{e_1\}$ and $\lambda_3 : A_3 \rightarrow \rho(H)$ such that $\lambda_3(e_1) = \{0, 1\}$,

$A_4 = \{e_1, e_2\}$ and $\lambda_4 : A_4 \rightarrow \rho(H)$ such that $\lambda_4(e_1) = \lambda_4(e_2) = \{0, a, 1\}$,

$A_5 = \{e_1, e_3\}$ and $\lambda_5 : A_5 \rightarrow \rho(H)$ such that $\lambda_5(e) = \begin{cases} \{b, 1\} & \text{if } e = e_1, \\ \{0, b, 1\} & \text{if } e = e_3. \end{cases}$

Then by routine calculations, we can see that, for any $2 \leq i \leq 5$,

$$(\lambda_i, A_i) \rightarrow \emptyset_\emptyset = \emptyset_\emptyset \text{ and } (\lambda_1, A_1) \rightarrow \emptyset_\emptyset = \emptyset_\emptyset \rightarrow \emptyset_\emptyset = H_E$$

Hence, it is clear that double negation property does not hold, and so $(\text{Soft}_E(H), \prod, \rightarrow, H_E, \emptyset_\emptyset)$ is not an MV -algebra.

Theorem 4.10. *Let $(\lambda, A), (\gamma, B) \in \text{Soft}_E(H)$. Define the relation \equiv on $\text{Soft}_E(H)$ as follows:*

$$(\lambda, A) \equiv (\gamma, B) \iff \exists(\varphi, C) \in \text{Soft}_E(H) \text{ st. } (\lambda, A) \prod(\varphi, C) \subseteq (\gamma, B)$$

Then \equiv is a congruence relation on $\text{Soft}_E(H)$.

Proof. It is clear that $(\lambda, A) \equiv (\lambda, A)$. If $(\lambda, A) \equiv (\gamma, B)$, then there exists $(\varphi, C) = (\gamma, B) \in \text{Soft}_E(H)$ such that $(\lambda, A) \prod(\gamma, B) \subseteq (\gamma, B)$. By the similar way, $(\gamma, B) \prod(\lambda, A) \subseteq (\lambda, A)$, and so $(\gamma, B) \equiv (\lambda, A)$. Hence, it is symmetric. Now, let $(\lambda, A) \equiv (\gamma, B)$ and $(\gamma, B) \equiv (\Delta, D)$. Since $(\lambda, A) \equiv (\gamma, B)$, there exists $(\varphi_1, C_1) \in \text{Soft}_E(H)$ such that,

$$(\lambda, A) \prod(\varphi_1, C_1) \subseteq (\gamma, B)$$

Also, from $(\gamma, B) \equiv (\Delta, D)$, there exists $(\varphi_2, C_2) \in \text{Soft}_E(H)$ such that,

$$(\gamma, B) \prod(\varphi_2, C_2) \subseteq (\Delta, D)$$

Then there exists $(\varphi_1, C_1) \prod(\varphi_2, C_2) \in \text{Soft}_E(H)$ such that

$$(\lambda, A) \prod(\varphi_1, C_1) \prod(\varphi_2, C_2) \subseteq (\gamma, B) \prod(\varphi_2, C_2) \subseteq (\Delta, D)$$

Hence, $(\lambda, A) \equiv (\Delta, D)$.

Now, let $(\lambda, A) \equiv (\gamma, B)$. Then there exists $(\varphi, C) \in \text{Soft}_E(H)$ such that, $(\lambda, A) \prod(\varphi, C) \subseteq (\gamma, B)$. Since \prod is the product operation on $\text{Soft}_E(H)$, for any $(\Delta, D) \in \text{Soft}_E(H)$, we get that,

$$(\lambda, A) \prod(\varphi, C) \prod(\Delta, D) \subseteq (\gamma, B) \prod(\Delta, D)$$

Thus, there exists $(\varphi, C) \in \text{Soft}_E(H)$ such that,

$$((\lambda, A) \prod(\Delta, D)) \prod(\varphi, C) \subseteq (\gamma, B) \prod(\Delta, D)$$

Hence,

$$(\lambda, A) \prod(\Delta, D) \equiv (\gamma, B) \prod(\Delta, D)$$

Moreover, suppose $(\lambda, A) \equiv (\gamma, B)$. Then there exists $(\varphi, C) = (\lambda, A) \prod(\gamma, B) \in \text{Soft}_E(H)$, such that $(\lambda, A) \prod(\varphi, C) \subseteq (\gamma, B)$. Let $(\Delta, D) \in \text{Soft}_E(H)$. Then

$$((\lambda, A) \rightarrow (\Delta, D)) \prod(\varphi, C) = \prod_{i,j \in I} \{(\lambda_i, A_j) | (\lambda, A) \prod(\lambda_i, A_j) \subseteq (\Delta, D)\} \prod(\varphi, C)$$

Since $(\text{Soft}_E(H), \prod, \coprod)$ is a distributive lattice, and $(\lambda, A) \prod (\varphi, C) \subseteq (\gamma, B)$. By considering $(\lambda_i, A_j) \prod (\lambda, A) = (\eta_i, F_j)$ we have,

$$\begin{aligned}
& \prod_{i,j \in I} \{(\lambda_i, A_j) | (\lambda, A) \prod (\lambda_i, A_j) \subseteq (\Delta, D)\} \prod (\varphi, C) \\
= & \prod_{i,j \in I} \{(\lambda_i, A_j) \prod (\varphi, C) | (\lambda, A) \prod (\lambda_i, A_j) \prod (\varphi, C) \subseteq (\Delta, D) \prod (\varphi, C) \subseteq (\Delta, D)\} \\
= & \prod_{i,j \in I} \{(\lambda_i, A_j) \prod (\lambda, A) \prod (\gamma, B) | (\lambda_i, A_j) \prod (\lambda, A) \prod (\gamma, B) \subseteq (\Delta, D)\} \\
\subseteq & \prod_{i,j \in I} \{(\lambda_i, A_j) \prod (\lambda, A) | (\lambda_i, A_j) \prod (\lambda, A) \prod (\gamma, B) \subseteq (\Delta, D)\} \\
= & \prod_{i,j \in I} \{(\eta_i, F_j) | (\gamma, B) \prod (\eta_i, F_j) \subseteq (\Delta, D)\} \\
= & (\gamma, B) \rightarrow (\Delta, D)
\end{aligned}$$

Hence, $(\lambda, A) \rightarrow (\Delta, D) \equiv (\gamma, B) \rightarrow (\Delta, D)$.

Now, let $(\lambda, A) \equiv (\gamma, B)$. Then there exists $(\varphi, C) \in \text{Soft}_E(H)$, such that $(\lambda, A) \prod (\varphi, C) \subseteq (\gamma, B)$. Suppose $(\delta, G) \in \text{Soft}_E(H)$. Thus,

$$((\delta, G) \rightarrow (\lambda, A)) \prod (\varphi, C) = \prod_{i,j \in I} \{(\delta_i, G_j) | (\delta_i, G_j) \prod (\delta, G) \subseteq (\lambda, A)\} \prod (\varphi, C)$$

Since $(\text{Soft}_E(H), \prod, \coprod)$ is a distributive lattice, $(\lambda, A) \prod (\varphi, C) \subseteq (\gamma, B)$. By considering $(\delta_i, G_j) \prod (\varphi, C) = (\eta_i, F_j)$ we have

$$\begin{aligned}
& \prod_{i,j \in I} \{(\delta_i, G_j) | (\delta_i, G_j) \prod (\delta, G) \subseteq (\lambda, A)\} \prod (\varphi, C) \\
= & \prod_{i,j \in I} \{(\delta_i, G_j) \prod (\varphi, C) | (\delta_i, G_j) \prod (\varphi, C) \prod (\delta, G) \subseteq (\lambda, A) \prod (\varphi, C)\} \\
\subseteq & \prod_{i,j \in I} \{(\delta_i, G_j) \prod (\varphi, C) | (\delta_i, G_j) \prod (\varphi, C) \prod (\delta, G) \subseteq (\lambda, A) \prod (\varphi, C) \subseteq (\gamma, B)\} \\
= & \prod_{i,j \in I} \{(\eta_i, F_j) | (\delta, G) \prod (\eta_i, F_j) \subseteq (\gamma, B)\} \\
= & (\delta, G) \rightarrow (\gamma, B)
\end{aligned}$$

Then $(\delta, G) \rightarrow (\gamma, B) \equiv (\delta, G) \rightarrow (\lambda, A)$. Therefore, \equiv is a congruence relation on $\text{Soft}_E(H)$. \square

Corollary 4.11. $(\text{Soft}_E(H) / \equiv, \prod, \rightarrow, H_E)$ is a hoop.

5 Conclusions and future works

In this paper, the concept of soft hoop is introduced and some examples are provided. Then, different types of intersections and unions of the family of soft hoops are established. We defined two operations \odot and \rightarrow on a hoop H and a parameters set E and we proved that the set of all soft hoops with these operations is a hoop and also is a Heyting algebra. Finally we introduced a congruence relation on the set of all soft hoops and we investigated the quotient of it. For the future works, it can be investigated that under which conditions, the set of all soft hoops can be considered as one of other algebraic structures.

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